Paul Dirac Talk: Projective Geometry, Origin of Quantum Equations
Audio recording made by John B. Hart, Boston University, October 30, 1972
Comments by Roger Penrose

[JOHN HART]

The following is Dirac's talk on projective geometry in physics at Boston University on October 30, 1972.

[INTRODUCTION SPEAKER]

It is impossible to introduce someone who is already known to everybody. Also it is true that most people here know who professor Dirac is, and there is no doubt that any branch of modern physics was either founded, or profoundly influenced by his work in the last 50 years or so. There is no need to dwell on the achievements of Professor Dirac, so I do not hold your patience any longer, and I will just mention that in his talk today, we'll hear more about informal orientation about certain feelings and ideas he has than highly technical talk. After Professor Dirac's talk, Dr. Penrose will take additional comments the subject, and of course the floor is then open for discussion. We are very grateful to Professor Dirac to have come here... [APPLAUSE]

[PAUL DIRAC]

When one is doing mathematical work, there are essentially two different ways of thinking about the subject: the algebraic way, and the geometric way. With the algebraic way, one is all the time writing down equations and following rules of deduction, and interpreting these equations to get more equations. With the geometric way, one is thinking in terms of pictures; pictures which one imagines in space in some way, and one just tries to get a feeling for the relationships between the quantities occurring in those pictures. Now, a good mathematician has to be a master of both ways of those ways of thinking, but even so, he will have a preference for one or the other; I don't think he can avoid it. In my own case, my own preference is especially for the geometrical way.

I may see a lot of great algebraic work and follow through all the steps of deduction, but still I find difficulty in grasping the significance of the whole thing. What I like to do, as far as possible, is to find a way of picturing the various quantities that I'll refer to in the algebra, and to try to express the algebraic relations in geometrical terms. There is, of course, a great deal of limitation on this; the algebra is just too complicated. Any kind of geometrical picture may be quite unworkable; there might be too many variables; too many to mention, so that it is quite hopeless to try to think of it. But usually so, there might be parts of the work which can be pictured geometrically, and I find that I can get a great deal of help by using these geometrical pictures whenever possible; the pictures bring out clearly, in my mind, the relationships between the quantities and point the way to getting further relationships.

So, I want, today, to talk to about the help I have received from geometrical methods in my work on physical theory. Now, in my published work, I have never referred to these geometrical methods. You might be inclined to infer from that that I just don't use these geometrical methods, but that would be wrong. The reason that I do not refer to these geometrical methods in published work is because it is awkward to publish it. If you publish geometrical figures, it means quite a lot of work for the publisher, also a lot of work for the person preparing the paper, setting up pictures which are suitable for printing, and it is much more convenient to put everything in algebraic terms, and to just publish instead some algebraic analysis. And so, I'm afraid that my published work is, in that way, a bit misleading as a guide to the kind of thoughts that I was using in my research work.

Now in the geometrical work, which I want to talk about, I put the emphasis on projective geometry. You may wonder why I do that. Why not just use the Euclidean geometry which we all learned about in school, which forms part of our basic mathematical training? It's quite logical, and we can understand it pretty well. Why don't we just keep to Euclidean geometry all the time? The reason for that is that projective geometry is more powerful. In Euclidean geometry, we learn a lot of theory which Euclid originated, but with projective geometry, we can get more powerful theorems with less work. It is a great deal of pleasure to use these more powerful methods, and of course to anyone who is lazy like me, there is quite a strong advantage in reducing the amount of work that you have to do to get the results. Once the basic ideas of projective geometry were understood by mathematicians, it completely superseded Euclidean geometry. When one has learned projective geometry, one doesn't want to go back to the clumsy Euclidean ways anymore.

So, projective geometry completely superseded Euclidean geometry, and for many decades it formed the basis of geometrical thinking. Well, projective geometry had its heyday and then gradually faded away. All the more elementary results were worked out and incorporated into textbooks, and there wasn't any new work for mathematicians to do, and so they went on to consider more general kinds of geometry, geometries of curved space, and more abstract even than curved space. Well, these more general geometries are not of much use to physicists. I guess the exception, of course, would be curved space that was introduced by Riemann, is necessary for understanding gravitational theory. All of Einstein's gravitational work is based on this kind of curved space, but gravitation plays a very insignificant role in physics. Atomic physics is concerned with particles interacting with forces and the gravitational fields are quite negligible, so that atomic physicists are working all the time with flat space. And if we're discussing flat space, the methods of projective geometry are the most powerful ones.

Now, I spoke a lot about projective geometry and I expect many of you don't know what it is. And I think that I should, first of all, try to explain to you the main ideas of projective geometry, try to get you to appreciate the power of projective geometry, how it is a much more agreeable kind of geometry to work with than Euclidean geometry. I shall do this general discussion taking a two-dimensional space, because that will enable me to present the ideas in the simplest way. Then, after this discussion on two-dimensional projective geometry, I would like to go on to the space which is of physical importance: the four dimensional space-time of Minkowski. And then, I shall show you how the projective ideas apply in Minkowski space.

Let us start then, with a two-dimensional space. As we think of all the theorems that we learned in two-dimensional space as schoolchildren, we learned about points and lines, triangles, circles, right angles, and things like that. Now, we might imagine our picture which gives us the understanding of geometrical theory to be projected according to the standard methods. We have our plane here; take a point here, project everything on the picture onto another plane over here, let us say. When carrying out such a projection process, of course, our figure gets distorted quite a lot, but some things remain invariant. A point remains a point, a line remains a line, when I say line I always mean straight line, and a triangle remains a triangle. However, a right angle does not remain a right angle, two parallel lines do not remain two parallel lines, a circle does not remain a circle, it goes over into a conic, hyperbola, or ellipse, or something like a parabola.

Now, in projective geometry, we just confine our attention to those features of the diagram which are unaltered by the projection process, and we just disregard all those features which get altered. That is to say, we talk about triangles, we talk about two lines being collinear, we can talk about three lines being conparent, three lines meeting at a point, but we cannot talk about a right angle anymore, because that ceases to be a right angle. We cannot talk about a circle anymore, because it ceases to be a circle, but we can still talk about conics. You see, we have lost a good many of the properties of Euclidean geometry, and you might think that we lost something important, but I will try to show you that we haven't really lost anything important. We might, perhaps consider the questions from the point of view of coordinates. We introduced coordinates into our Euclidean space; we had each point represented by two coordinates, x_1 and x_2, say, and we make transformations of coordinates, so this time, $x_1^* = a_{11}x_1 + a_{12}x_2 + b_1$, and $x_2^* = a_{21}x_1 + a_{22}x_2 + b_2$. And we're not interested in the geometrical theorem, we are interested in relations between points and lines, which are undisturbed by such a transformation of coordinates.

Now, if we are dealing with the subject according to projective geometry, each point is represented not by two coordinates, but by three coordinates, and it is only the ratios of these three coordinates which are important. Each point is now represented by three coordinates, y_1, y_2, y_3, and if you change all three coordinates with the same ratio, they still refer to the same point. You can take any numerical multiplying factor ρ, ρy_1, ρy_2, ρy_3, our coordinates are the same point as y_1, y_2, y_3, and when we make a transformation of coordinates, the transformation behaves like this: $y_r^* = c_{rs}y_s$ for all values of s. The transformation is just a homogeneous transformation of the three coordinates. The three coordinates are treated symmetrically. We can make any transformation which we like of this type, subject to the condition that the determinant of these coefficients does not vanish so that we can invert the transformation, and that would be a permissible transformation of the coordinates into our two-dimensional projective space.

When you see that the transformations are really much simpler in the projective geometry, all transformations of this kind are treated together to form a very simple group of transformations. Here we have these rather awkward transformations, we've got these coefficients b_1 and b_2 sticking out, standing out differently from the others and we have to impose some special conditions on these coefficients. Of course, we might use three coordinates and not bother about those conditions, but three coordinates are usually not so convenient in Euclidean geometry.

Now, what is the connection between these two ways of describing a space (the two-dimensional space)? We get the connection if we consider these ratios: $x_1 = y_1 / y_3$, $x_2 = y_2 / y_3$. In this projected space, where we are interested only in the ratios of the three coordinates, we might turn our attention to these ratios here. In doing so, of course, we are spoiling the symmetry between the three y's, but we are getting somewhere nearer the Euclidean space. These x_1 and x_2 can now play the role of x_1 and x_2 in the Euclidean space. We should do this transformation; otherwise it becomes the non-homogeneous transformation of these two x's. When you look at it from that point of view, the coordinates of the Euclidean space seem to be a rather awkward way than these more symmetrical arrangements in the projective space.

If we take a line that would correspond to a linear equation between x_1 and x_2 (in the projective space l_1x_1 and l_2x_2), that is the equation of a line in the Euclidean space. [SHOWN ON BOARD] The equation of a line in the projective space would be anywhere near equation connected to the three y's. Let us say $n_1y_1 + n_2y_2 + n_3y_3 = 0$. Here again, you see something which is more unique because it is homogeneous. It is neater than this description of a line in the Euclidean space. Now, every line here corresponds to a line here, if we just identify the x's, x_1 and x_2, as ratios of the y's. However, the converse is not true.

It is not true that every line in the projective space corresponds to a line in the Euclidean space. There is one line here, namely, the line of the equation $y_3 = 0$. That does not correspond to any line at all in the Euclidean space. Now, it's a bit awkward having an exceptional line in the Euclidean space, which makes the relationship between the Euclidean space and projective space rather awkward to describe. What we do is we say that there is a line in the Euclidean space corresponding to this line in the projective space. You can call it the line at Infinity in the Euclidean space. Euclidean space thus becomes projective space in which there is one line which is singled out and is called the line at Infinity. While in the projective space, all the lines are treated on the same footing. The Euclidean space is beginning to appear something a bit artificial. Now, how about a circle? Beginning with a circle in Euclidean space, you could have an equation something like this: $x_1^2 + x_2^2 + 2ax_1 + 2bx_2 + h = 0$. That's the equation of a circle.

Now, in projective space, we don't have circles, we have conic sections or conics as I recall them briefly. And a conic is described by an equation which is homogeneous, and of the second degree in the three coordinates. We have a conic described by σ_{rs} + $e_{rs} + x_r + y_r + y_s = 0$. Now, if we suppose that the y's determine the x's (the ratios of the y's determine the x's)... [INAUDIBLE] Then, any circle gives us the homogeneous equation between the y's this time, but it is a homogeneous equation of a special kind. We may take the point with the coordinates $y_1 = 1$, $y_2 = i^2 - 1$, $y_3 = 0$. Let's call this point I. This point has imaginary coordinates, but we need not let that worry us at all. It's just a point according to the standard meaning of a point, as with any other point, and you see that if you put down the equation between the y's: $y_1^2 + y_2^2 + 2ay_1y_3 + 2by_1y_3 + h = 0$. You just put the x's equal to the ratios of the y's and follow this scheme here, you get this, which is a conic in the projective space and this conic has the property that this point I lies on the conic. If you put $y_1 = 1$, $y_2 = \sqrt{-1}$, $y_3 = 0$, then this equation is satisfied.

Then take another point, called J, where $y_1 = 1$, $y_2 = -i$, $y_3 = 0$, which also satisfies this equation. The result is, that you have here a conic in our projective space and it has two points, I and J, which lie on the conic. We now get a new understanding of circles. Circles are just the conics in projective space which passed through these two points, I and J. We have our projective space here. Take any line, call it the line at Infinity, take any two points on the line, I and J, and then any conic that goes through those two points I and J, will correspond to a circle. There's really nothing special about a circle to distinguish it from other conics, so far as concerns projective geometry, and if we want to get over to Euclidean geometry, we mix the circle with something special, we have to choose two points in our projective geometry and label them I and J.

They are two points on the line at Infinity, and when we agree to take these two points I and J, we get over to a Euclidean space in which every conic going through I and J gets counted as a circle. Well, there we see how some of the important ideas of Euclidean geometry get translated into ideas in projective geometry. In Euclidean geometry there are other ideas which we talked about, for example, two lines being parallel. Two lines being parallel means that they intersect at a point on the line at Infinity. Parallel lines would correspond to parallel lines in the Euclidean picture. Now, another important idea in Euclidean geometry is perpendicular lines. How are we to understand perpendicular lines in projective geometry? Well, there is an important idea in projective geometry about two points which harmonically separate two other points. It is explained in projective geometry. If we take any line here, and any point A or B on the line, now, we take another point, P. Then, we set up this construction: draw two lines through A, like this, take any line through B which intersects those first two lines and leads to a point, here and here, then join these two points to B. Then you get two new points; join these two new points, and we get to a point out here, which we call Q.

We have here a simple construction in which we started from two points, A and B, another point P, and we get another point, Q. And it is quite easy to prove all ways of doing that construction will lead to the same point Q. You may take these two lines in various ways, two lines through A, then a line through B, any way that we like, do the necessary joining up, and we always finish up with the same point Q. This point, Q, is related in some special way to these other three points, and we say P and Q harmonically separate A and B. The relation is a symmetrical one; A and B also harmonically separate *P* and *Q*. Now, let's go back to this picture here. Suppose we take two lines, they meet this time at Infinity, and two points which are harmonically separated by *I* and *J*, and those two lines correspond to two perpendiculars in the Euclidean space. So, in that way we can introduce the idea corresponding to perpendicularity in the projective space. There is a little further development which one can make for something corresponding to distance into the projective space, but I don't think I need to enter that; it's not of such general importance.

But the situation now is that when we have any picture in the projective space, we take two points, label them *I* and *J*, and the line joining them, we label the line at Infinity, and in the end we get a picture in Euclidean space. Conversely, any picture in Euclidean space will give us a picture in projective space a picture which involves these two special points, *I* and *J*. From the point of view of projective geometry, the points *I* and *J* are no different from any other points. Euclidean geometry just, so to speak, picks out two points in the projective space and assigns to them special properties. These two points and the line joining them are called the absolute, they are assigned special properties, this assignment is really quite arbitrary, and when we do this assignment, we get a picture in Euclidean space.

We're now in the position to be able to take any of Euclid's theorems and translate it into a picture in projective geometry. I would just like to give you a simple example to show you how it goes. I'll take this example: if you take the diameter of a circle and any points *A* and *B*, and draw lines from *A* to *E* and *B* to *E*, these two lines are at right angles, something we all learned as schoolchildren. We can now translate this at once into a theorem of projective geometry. What we have now, is this diagram.

This circle becomes a conic; let's draw it as an ellipse. Then, this conic goes to two points, *I* and *J*; let's draw them out here, call this one *I*, this one *J*. Now, here we have the center of the circle, let us say, *C*. What point does *C* correspond to in this picture in projective space? Well, one can work that out quite easily and the answer is that you take tangents to the circle at the points *I* and *J* and see where they meet. Where they meet is the center of the circle, *C*. Now, you might say that doesn't look right [LAUGHTER]; the center is not even inside our conic, and the reason why it doesn't look right, is that we've drawn these points, *I* and *J*, as real points when really they're imaginary. That, of course, puts the picture altogether out, but the arguments which we make this picture remain correct. And so that here, we have these points going through *I* and *J*, and point *C* really is the center of the circle.

Now, the diameter *AB*, that's just a straight line going through the center, so we pick any line through *C*, such as this one, and where it meets the circle, we get the two points *A* and *B*. Then, we take any other point on the circle, say point *B*, join it up like this, *BA* , *BE*, and then -- it's not really convenient as I've drawn it here, I'm going to have this point go off the board -- take point *B* to be here. Draw *BA*, *BE*, and then, we see that *BE* and *BA* harmonically separate *I* and *J*. The condition for the lines *BA* and *BE* to be perpendicular to each other, which is what this theorem of Euclid tells us, are harmonically separated by *I* and *J*. There you see a well-known theorem of Euclidean geometry becomes a theorem about ellipses; actually, a much more general theorem.

You might say you're not very much interested in ellipses, why bother about them, but you see what you can do. Once you've got this projective theorem, you can take any other two points and call them *I* and *J*, if you like. There's nothing really special about these two points; you might call them points *A* and *B*, set up a new picture where *A* and *B* become *I* and *J*, and then you would get a new Euclidean picture describing a new theorem. But it turns out in this case that the new theorem which you get is practically identical to the original one, so you haven't really gained anything. But in other examples, you do get a new theorem by taking two different points and calling them *I* and *J*. You see that to understand the relationship properly, you ought to work without calling any two points *I* and *J*, to be perfectly general, and then you have a projective picture.

Now, there's another very interesting procedure that you can make in projective geometry, using the equation of a line. I wrote it up there: $n_1y_1 + n_2y_2 + n_3y_3 = 0$. The line is described by the three *n*'s. It's only the ratio of the *n*'s that matter; the three *n*'s can be looked upon as the coordinates of a point in another projective space. We can get, in that way, a second projective space in which every line of the first projective space corresponds to a point in the second projective space; any point of the first projective space corresponds to a line in the second projective space. We get a new projective space in which the concepts of point and line are interchanged. That's something we can't do in Euclidean geometry, but it means that, when we've got any theorem in projective geometry, such as the one that I've described here, you can get another theorem by replacing all the points by lines and all the lines by points.

The purely mechanical procedure will enable you to get a new theorem starting with any particular theorem. I would just like to mention what happens inside this process, which is called the process of taking a dual, for this example here. The conic remains a conic. Points on the conic, become lines touching the conic. We get a new picture in which we have a conic like this: all these points on the conic become lines touching the conic, if I could draw them properly...lines *I* and *J*. I will use small letters to denote the lines corresponding to the points in this picture. We have these lines, *I* and *J*, touching the point corresponding to these two points, *I* and *J*, on the conic. Then, we take some more lines, *A* and *B*, this one corresponding to *A*, this one corresponding to *B*. then we take one more line, let's say the line *B*, I'll draw it out here. This is the line *B* corresponding to this point *B* here.

Now, here we have the line *AB*; here we shall have corresponding with a point where *A* and *B* intersect; that will be down here; this is the point here. Then, we take *B* that we have here, and then we have the theorem that tells us that these two points *A* and *B*, are harmonically separated by these two lines, *I* and *J*. This point, and this point, are harmonically separated by this point, and this point right here. We've got another theorem, which is essentially the same theorem, the general one, that every point has become a line, and every line has become a point. Now, we may take this new theorem which we've got here, take any two points and it, call them *I* and *J*, and get back to the theorem in Euclidean space. What do we get, if we take for example...we want to take these two points, *I* and *J*, then our conic again becomes a circle, and we get a theorem in Euclidean geometry, let me figure it out... [SEARCHING]...I know I've got the theorem in my notes...this is the line *B*, this is *A*, this is *B*... [WRITING ON BOARD]...this is the point *IJ* where the two lines *I* and *J* intersect.

The theorem now tells us that these two lines are perpendicular to each other. Then you take two parallel lines, *A* and *B*... We now get a rule which tells us that if we have two parallel lines touching a circle, then take any third line like this touching the circle, and draw these two lines from the center, then these two lines are perpendicular to each other. We've got a new theorem in Euclidean geometry starting out from the original theorem just by a mechanical procedure, a sort of procedure which a computer can do. We do that by crossing over to projective space, and then projecting projective space into Euclidean space in various ways. Well, I think that this will give you some idea of the power of projective geometry. It is essentially a device for saving work, for enabling you to do in a single piece of work, what would take several different pieces of work if you just followed by Euclid's methods, and once you've gotten familiar with these ideas of projective geometry, you never want to go back to Euclid's methods again.

Well, that is the discussion of two-dimensional projective space, but there is, of course, a lot of details to fill in before one would master the subject. I would like now to go onto something which is of more value to the physicists. A physicist is not very often concerned with two-dimensional space. Suppose you go over to three-dimensional space. In three-dimensional Euclidean space, each point has three coordinates which transform non-homogeneously. In the three-dimensional projective space, each point will be specified by the ratios of all coordinates, and the four coordinates transform homogeneously.

In the projective space we must take a certain line...in the projective space, we must take a certain plane and call it the plane at Infinity, if we want to get a connection with the three-dimensional Euclidean space. And then instead of the two points, *I* and *J*, the two points forming the absolute in two-dimensional space, we have a conic in this plane and we call that conic the absolute, and we say that any quadric -- a quadric is a [INAUDIBLE] of sorts, represented by a homogeneous equation, with what would be four variables in the three-dimensional projective space. Any quadric which meets this plane at Infinity in this absolute conic, is counted as a sphere. Then we have a similar way of taking care of perpendicularity. Then, we can go on from that to four dimensional space. Now, in four dimensions we need five homogeneous coordinates which will have an absolute now which consists of four dimensional flat space, corresponding to the space at Infinity, and we shall have a quadric in this space at Infinity playing the role of the absolute.

Now, as physicists where interested in not four dimensional Euclidean space, but four dimensional Minkowski space, and that will result in the absolute quadric becoming a real quadric instead of an imaginary one. In all this discussion of Euclidean space going over to projective spaces, we always had an absolute involving imaginary points.

Although we can argue about them, when we draw pictures they seem to be wrong. When we go over to Minkowski space, the absolute becomes a real quadric, and that makes it very much easier to understand. Perhaps it's very special and fortunate, the circumstance that physics is interested in Minkowski space, and Minkowski space is an especially favorable one to understand from the point of view of projective geometry. We have an absolute now, consisting of a three-dimensional space and a quadric in this three-dimensional space.

Now, a lot of the work in physics involves just the directions of vectors and the directions of various quantities, and if we're discussing the directions, it is sufficient just to see where these directions meet the hyper plane at Infinity; they will meet the hyper plane at Infinity at certain points, lines, or whatever structure it is we are interested in. And if we just think in terms of this hyper plane at Infinity, we have a three-dimensional space. Talking of a four dimensional space is something that is hard to imagine, but we can't really imagine it. We talk about it as though we could, but when we are concerned just with directions, the things in the space of physics, we can represent them all in terms of a three-dimensional space according to the methods of projective geometry. We have a three-dimensional projective space in which there is an absolute quadric. We imagine this quadric is a sort of ellipsoid, and any direction in the physical space corresponds to a point in this three-dimensional projective space.

We see, at once, that there are three kinds of points: points that lie inside the ellipsoid, points that lie outside the ellipsoid, and points that lie on the ellipsoid. Now, those correspond to the three kinds of directions which we have in physical space. Points inside the quadric correspond to the direction of the particle-- timeline direction; points outside the quadric correspond to space line directions; points on the quadrant correspond to the directions of light rays, or null directions, as you might call them. These directions are all immediately pictured in the three-dimensional projective space, and all the other relationships between directions of things in physical space can immediately be [TAPE SKIPS] in physical space.

If we have three directions, such that the superposition of two of them is to the third, then that would correspond to three points in the projective space which are collinear. One result which you can immediately see goes like this: suppose we take two points, say *A* and *B*, in our three-dimensional projective space, which I've always done by writing like this. This time, we'll have the points meet the quadric [INAUDIBLE]... Now, here we have four points on a straight line, and four points on a straight line can have their relationship described by an invariant number, a number which is invariant under projections called the cross ratio of those four points. That corresponds to the fact that these two points, *A* and *B*, have an invariant angle between them and we think of them as two points, two directions in physical space. But, it holds also that these two points, *A* and *B*, are outside the quadric, or maybe one inside and one outside, and we can then join them up if these two points, *P* and *Q*.

Now, it might be that this time it does not meet the quadric; that's not to say it does not meet the quadric, it's just something that would be described mathematically by saying it meets the quadric at two imaginary points, just lying outside, like this, and we still get the four points such that we can form an invariant number from them. However, if this line should touch the quadric, then we only have three points. That means that if we have two directions in space-time, such that the line joining them in this picture is [TAPE SKIPS] nowhere near between those two directions. That is something, of course, which you could work out, with a little bit of algebra, but you see it at once on the geometrical picture. Now, I would like to discuss a few more things in physics, and show how they correspond to various things in this three-dimensional projective space.

We are continually dealing with directions which are orthogonal to each other. Orthogonal means essentially perpendicularity, but we use the word orthogonal when talking about Minkowski space; to replace the word perpendicularity is rather reserved for Euclidean space. If we have two points which are orthogonal to each other, they will correspond to two points in this picture which are related in a special way. That special way is that polar angle of one of them crosses through the other. If this is one of the points, and we make these tangents here, all of these tangents lie on the plane, all of these tangents cross the plane, which is called the polar plane that we started with, and if we take a second point, *P*, and a second point, *Q*, lying on the polar plane, *P* and *Q* correspond to two directions that are orthogonal. It's possible to have this orthogonal relationship holding also that both *P* and *Q* are both outside the [INAUDIBLE].

If we want to get an orthogonal system of axes, such as physicists are always using, we have to take a four point space; just one of them would be inside the quadric, the other three would be outside, and any two of them have to be orthogonal in accordance with this polar-polar relationship. Now we're dealing with Lorentz transformations all the time in physics. What might a Lorentz transformation correspond to in this picture? It corresponds to a linear transformation of our coordinates, the four coordinates of this three dimensional projective space, a linear combination of this transformation, we could just say a projective transformation, and it is such that this quadric remains invariant. Any Lorentz transformation can thus be pictured as a projective transformation in three dimensions, which leaves the quadric invariant. That is quite a good way of picturing a Lorentz transformation.

Now, when people are studying Lorentz transformations, they often want to confine their attention to a certain subgroup of them, rather than the whole group of Lorentz transformations, and this, of course follows immediately in terms of this picture. We want to take the subgroup of Lorentz transformations which leaves a certain time axis invariant. That can mean all projective transformations which leave the quadric invariant, and also leave this point invariant. Now, suppose we just want to consider Lorentz transformations which leave a certain time axis invariant, and can also leave one of the space axes invariant, let us say the axis *t* and the axis *z*. That would mean that this point here, corresponding to the time axis, and another point here, corresponding to the *z*-axis, which is left invariant; these two points are left invariant, the quadric also has to be left invariant, and thus, these two points are left invariant. It means that every point on this line is invariant. If the four points on the line are invariant, then every point has to be invariant.

So, we are making a projective transformation which leaves the quadric invariant, and leaves every point on this line invariant. Now this line must be a polar line up here, polars are all the points here that pass through a line; this line would be a polar line. This polar line will have to remain a polar line, because it is connected in an invariant way to this line with the quadric, but the points on this polar line will get changed. This polar line is subject to transformations; they will just be a single parameter family of transformations, which will correspond to the rotations which leave the time axis invariant and also leave the *z*-axis invariant. But, here we see a geometrical way of studying polar relations, and we see there are corresponding ways of generalizing. Instead of taking those Lorentz transformations, which leave all the points along this line invariant, we can take another line which touches the quadric, and consider Lorentz transformations which leave every point at this line invariant. Well, we can figure out just what those Lorentz transformations are like and we get results by the geometrical methods which are much more direct and easy to understand, than if we used the direct elliptical approach. In this case, there's really not much advantage in using a projective picture, because we're all familiar with that kind of Lorentz transformation-- we just make the rotation about an axis.

But here we have some transformations which are not so familiar, and we're going to figure out the relationships more directly with the help of these ideas of projective geometry. Well, I don't know how long I should go on speaking; I'll say a few more things, perhaps. Suppose we want to consider an angular momentum in Minkowski space. Well, if we have a particle moving about a center, it would have a definite angular momentum. We shall have in our projective space, a point here represented with x's, which we are [INAUDIBLE]... and we have a point here to represent momentum, and another point to represent rotation, [WORKING ON BOARD]... the points I gave you, the points *B* and *x*, would be represented by two points in this projective picture, and the angular momentum would be represented by the line drawn here.

Angular momentum corresponds to a line in our projective picture, but this line is not a general angular momentum. If we had this one angular momentum, and then take another angular momentum, and add it on, we begin to get something of a different nature-- a rather more complicated nature-- something which is not representable by a line in this projective picture, something that the geometries are quite familiar with, it's called a linear complex, a whole set of lines, which sets a certain kind of linear condition on their coordinates.

I think that's enough discussion on the details of this. You might say that you find these projective ideas especially useful in discussing plane electromagnetic waves, in which we have a set of plane waves, and each of them will need four parameters to describe it: three parameters to describe the direction of motion of the waves, and then a fourth parameter to describe its location in time. We may have a whole set of waves going in the same direction and they may differ only in the fourth parameter. Well now, according to this principle of duality which I was talking about, the plane wave, which I may call a front of these waves, can be represented by a point. It's just the ratio of these four coordinates that matter. We get a space to represent the fronts, and this space is of the nature of the three-dimensional space in four dimensions, forming a sort of a cylinder. Of course, it's hard to draw a picture of a cylinder in four dimensions. In order to do it, you kind of have to drop out a few dimensions. If you do drop them out, then you get a normal cylinder in three-dimensional space. A cylinder like that will represent all the fronts of all the waves traveling with the velocity of light which we have in physical theory.

Suppose we want to consider those waves which pass through a definite point in space-time. That would correspond to points in this cylinder picture, which all lie on the plane. Well, I think I'll stop at that point, as I've shown you how physical ideas can be pictured immediately in geometrical terms, and the projective geometry is especially convenient for discussing Minkowski space. And, I hope the earlier part of my talk enabled you to appreciate the power of projective geometry and how it is really much more fun working with projective geometry than with Euclidean geometry. [APPLAUSE]

[ROGER PENROSE]

The following is a quote from Dr. Roger Penrose's closing remarks.

"One particular thing that struck me... [LAUGHTER]...is the fact that he found it necessary to translate all the results that he had achieved with such methods into algebraic notation. It struck me particularly, because remember I am told of Newton, when he wrote up his work, it was always exactly the opposite, in that he obtained so much of his results, so many of his results using analytical techniques and because of the general way in which things at that time had to be explained to people, he found it necessary to translate his results into the language of geometry, so his contemporaries could understand him. Well, I guess geometry… [INAUDIBLE] not quite the same topic as to whether one thinks theoretically or analytically, algebraically perhaps. This rule is perhaps touched upon at the beginning of Professor Dirac's talk, and I think it is a very interesting topic."